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On the Number of Absolutely Indecomposable Representations of a Quiver

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A conjecture of Kac states that the constant coefficient of the polynomial counting the number of absolutely indecomposable representations of a quiver over a finite field is equal to the multiplicity of the corresponding root in the associated Kac–Moody Lie algebra. In this paper we give a combinatorial reformulation of Kac's conjecture in terms of a property of q-multinomial coefficients. As a side result we give a formula for certain inverse Kostka–Foulkes polynomials.

Key Words: Hall algebra; symmetric functions.

1. INTRODUCTION

Throughout Q will be a fixed quiver without loops (although this is not very essential). We denote the vertices of Q by Q_0. For a given $\alpha \in \mathbb{N}^{Q_0}$ let $o_\alpha(q)$, $i_\alpha(q)$, and $a_\alpha(q)$ be, respectively, the number of representations of Q over $\mathbb{F}_q$ with dimension vector α (recall that an indecomposable representation is said to be *absolutely indecomposable* if it remains indecomposable after extension of the base field).

In [12] Kac proves the following.

Theorem 1.1. *$o_\alpha(q)$, $i_\alpha(q)$, and $a_\alpha(q)$ are polynomials in q. $o_\alpha(q)$ and $a_\alpha(q)$ have integral coefficients.*

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Computing some examples lead Kac to two conjectures regarding $a_\alpha(q)$.

Conjecture 1.2 [12]. $a_\alpha(q)$ has positive coefficients.

This conjecture is still open. Some positive results are offered in [10, 15, 19]. These papers also yield evidence for the conjecture that $o_\alpha(q)$ has positive coefficients.

If true the conjecture would yield some evidence for the hope that absolutely indecomposable representations can be parametrized by a union of affine spaces. However proving the latter seems to be totally out of reach of current techniques. See [1, 2, 8, 9, 16] for results on the generic version.

However even without the existence of such a "cell decomposition" one has the feeling that the coefficients of $a_\alpha(q)$ should have an interpretation in terms of representation theory. Such an interpretation is offered for the constant term by Conjecture 1.4 below, which is also due to Kac.

To state this conjecture we have to introduce more notation. Recall that associated to Q there is a bilinear form on $\mathbb{C}^{Q_0}$ given by

$$(e_i, e_j) = \begin{cases} 2 & \text{if } i = j \\ -a_{ij} - a_{ji} & \text{otherwise}, \end{cases}$$

where $(e_i)_j = \delta_{ij}$ and where a_{ij} is the number of edges going from i to j.

Let $\mathfrak{g}$ be the Kac–Moody Lie algebra associated to the bilinear form $(-,-)$. The following result is proved in [11, 12] (see also [14]).

THEOREM 1.3. *If α is a real root of $\mathfrak{g}$ then $a_\alpha(q) = 1$. If α is an imaginary root then $a_\alpha(q) \neq 0$. If α is not a root then $a_\alpha(q) = 0$.*

This last theorem provides some motivation for the following conjecture, which, as was already mentioned above, is also due to Kac.

Conjecture 1.4. $a_\alpha(0)$ is equal to the multiplicity of α in $\mathfrak{g}$.

This conjecture is trivially true in the Dynkin case (in view of Theorem 1.3), and using the known representation theory of tame quivers [5], one can show that it is also true in the extended Dynkin case. However it is not known for a single wild quiver. On the other hand there is a lot of positive computer evidence. For example we have checked with little effort (using Theorem A below) that for the 3-arrow quiver

$$\begin{array}{ccc} & \rightarrow & \\ o & \rightarrow & o \\ & \rightarrow & \end{array}$$

the conjecture holds up to dimension vector $(20, 20)$.

Our aim in this note is to reformulate Conjecture 1.4 into a property of Gaussian multinomial coefficients. This is of theoretical interest since such multinomial coefficients are known to have a lot of non-trivial combinatorial properties. Furthermore as was already indicated above, our reformulation is considerably easier to verify by computer than the original statement of Conjecture 1.4.

For an entirely different approach to Conjecture 1.4 through the Hall algebra [18] we refer to [20].

To give our reformulation of Conjecture 1.4 we have to introduce some more notation. One easily verifies that the linear transformations

$$s_j\colon \mathbb{C}^{Q_0} \to \mathbb{C}^{Q_0}\colon \alpha \mapsto \alpha - (\alpha, e_j)e_j$$

have order 2 and leave $(-,-)$ invariant. The group generated by the $(s_j)_j$ is called the Weyl group of Q and is denoted by W_Q. As usual if $w \in W_Q$ then $\epsilon(w)$ is ± 1 depending on whether w is a product of an even or odd number of (s_j)'s. Following [13] we also introduce a formal symbol ρ and we extend the action of W_Q to $\mathbb{C}^{Q_0} \oplus \mathbb{C}\rho$ by putting $s_i\,\rho = \rho - e_i$.

If $a, b_1, \dots, b_n \in \mathbb{N}$ are such that $\sum_i b_i = a$ then the corresponding Gaussian multinomial coefficient is defined by

$$\begin{bmatrix} & a & \\ b_1 & \cdots & b_n \end{bmatrix} = \frac{\phi_a(t)}{\phi_{b_1}(t)\cdots\phi_{b_n}(t)},$$

where $\phi_a(t) = (t^a - 1)(t^{a-1} - 1)\cdots(t-1)$. If $\mu = (\mu_1, \mu_2, \dots)$ is a partition then we put

$$[\mu] = \begin{bmatrix} \mu_1 \\ \mu_1 - \mu_2, \mu_2 - \mu_3 \ \cdots \end{bmatrix}.$$

By definition a *multipartition* of $\alpha \in \mathbb{N}^{Q_0}$ is a list of partitions $\lambda = (\lambda_i)_{i \in Q_0}$ such that $|\lambda_i| = \alpha_i$. We view λ as a two dimensional list of non-negative integers $(\lambda_{ij})_{ij}$ such that $\sum_j \lambda_{ij} = \alpha_i$. We write $\lambda(j)$ for the element of $\mathbb{N}^{Q_0}$ given by $\lambda(j)_i = \lambda_{ij}$. Denote the set of multipartitions of α by $\mathscr{P}_\alpha$.

THEOREM A. *The Laurent polynomial*

$$p_\alpha(t) = \sum_{\lambda \in \mathscr{P}_\alpha} t^{\sum_j (\lambda(j), \lambda(j))/2} \prod_{i \in Q_0} (-1)^{\lambda_{i1}} t^{-\lambda_{i1}(\lambda_{i1}+1)/2} [\lambda_i]$$

is contained in $\mathbb{Z}[t^{-1}]$. *Furthermore, Conjecture* 1.4 *is equivalent to the statement that*

$$p_\alpha(\infty) = \begin{cases} \epsilon(w) & \text{if } \alpha = \rho - w\rho, \text{ where } w \in W_Q \\ 0 & \text{otherwise}. \end{cases}$$

For example in the case of the m-arrow quiver

$$o \quad \begin{matrix} \rightarrow \\ \vdots \\ \rightarrow \end{matrix} \quad o,$$

$p_{a,b}(t)$ would be given by

$$\sum_{\substack{|\lambda|=a \\ |\mu|=b}} (-1)^{\lambda_1+\mu_1} t^{\sum_i \lambda_i^2 + \sum_i \mu_i^2 - m\sum_i \lambda_i \mu_i - \lambda_1(\lambda_1+1)/2 - \mu_1(\mu_1+1)/2} [\lambda][\mu]. \quad (1.1)$$

For the convenience of the reader let us recall one of the standard combinatorial interpretations of Gaussian multinomial coefficients. Let $a, b_1, \dots, b_n \in \mathbb{N}$. Assume that $u_1, \dots, u_n$ are symbols. The cardinality of the set $\mathscr{W}$ of all words in the u_i with u_i appearing exactly b_i times is given by the ordinary multinomial coefficient $a!/b_1! \cdots b_n!$. A map $\phi: \mathscr{W} \to \mathbb{N}$ such that

$$\sum_{w \in \mathscr{W}} t^{\phi(w)} = \begin{bmatrix} & a & \\ b_1 & \cdots & b_n \end{bmatrix}$$

is called a Mahonian statistic [4, 7].

There are many Mahonian statistics [4, 7]. One example is given by

$$\mathrm{INV}(u_{i_1} \cdots u_{i_a}) = \left|\{ i_u > i_v \mid u < v\}\right|.$$

After choosing a Mahonian statistic, the evaluation of $p_\alpha(\infty)$ amounts to a signed counting of words. Following an established technique in combinatorics it should now be possible to define an involution on these words which inverts almost all signs. It seems very likely that in this way it should be possible to obtain a proof of Conjecture 1.4 but unfortunately we have not succeeded in completing this program.

Although Theorem A was meant as a possible step in the proof of Conjecture 1.4 it also has independent interest. For example, it is possible to use it in order to obtain some new identities on Gaussian multinomial coefficients. This is outlined in Section 6.

In our proof of Theorem A we needed the entries $K^{-1}_{\lambda, 1^n}(t)$ of the inverse Kostka–Foulkes matrix in one variable. A combinatorial interpretation for $K^{-1}_{\lambda,\mu}(1)$ was given in [6] but the methods used in that paper do not seem to extend to the one-parameter case. Using another method instead, based on a recurrence relation (A.3) derived from Pieri's formula, we obtain the following result.

THEOREM B. *Assume that* $\mu = (c, 1^d)$ *is a hook. Then*

$$K^{-1}_{\lambda,\mu}(t) = (-)^{l(\lambda)+l(\mu)} t^{\sum_{i \ge 2} \lambda'_i(\lambda'_i+1)/2 - \sum_{j=2}^{c} \lambda'_j} [\lambda'] \frac{1 - t^{\lambda'_c}}{1 - t^{\lambda'_1}}. \tag{1.2}$$

Our proof of this result is reproduced in Appendix A. We were informed by Richard Stanley that in the case relevant to us ($\mu = 1^n$) (1.2) was conjectured by Carbonara, and subsequently proved by Macdonald, using a different method [3]. It is not clear to us that Macdonald's method extends to the more general case of hooks. On the other hand one of the referees of this paper indicated a different (and ingenious) proof of (1.2) which is also valid for hooks, but which uses somewhat more sophisticated machinery than our naive proof.

We wish to thank V. Kac for commenting on a first version of this paper. In particular he informed us that the Ph.D. thesis of Jiuzhao Hua [10] contains related results. We also wish to thank Stanley for showing us his very interesting notes on the generating function of $o_\alpha(q)$ [21]. Both Stanley's notes and Hua's thesis contain (in a different notation) the rational functions $r_\alpha(q)$, defined herein.

2. A RECURRENCE RELATION FOR $o_\alpha(0)$

Let k be an arbitrary field. A quiver Q is a quadruple (Q_0, Q_1, t, h), where Q_0 and Q_1 denote, respectively, the vertices and the arrows of Q, and h, t: $Q_1 \to Q_0$ are maps associating an arrow with its head and tail. We assume throughout that Q_0, Q_1 are finite sets and that our quivers have no loops.

A representation V of Q is a pair $((V_i)_{i \in Q_0}, (\phi_e)_{e \in Q_1})$, where the V_i are finite dimensional vector spaces and the ϕ_e are maps $V_{t(e)} \to V_{h(e)}$. The element of $\mathbb{N}^{Q_0}$ given by $(\dim V_i)_i$ is the *dimension vector* of the representation. Homomorphisms between representations are defined in the usual way.

Define $R(Q, \alpha, k) = \prod_{e \in Q_i} \operatorname{Hom}(k^{\alpha_{t(e)}}, k^{\alpha_{h(e)}})$ and $GL(\alpha, k) = \prod_{i \in Q_0} GL(\alpha_i, k)$. Every element $x \in R(Q, \alpha, q)$ defines a representation V_x of Q. It is clear that $GL(\alpha, k)$ acts on $R(Q, \alpha, k)$ by conjugation of matrices and furthermore $V_x \cong V_y$ if and only if x, y are in the same $GL(\alpha, k)$-orbit. Thus the isomorphism classes of representations of dimension vector α are in one–one correspondence with the orbits $R(Q, \alpha, k)/GL(\alpha, k)$.

From now on k will be a finite field with q elements. We write $R(Q, \alpha, q)$ for $R(Q, \alpha, k)$ and $GL(\alpha, q)$ for $GL(\alpha, k)$. For $\alpha \in \mathbb{N}^{Q_0}$ let $U(\alpha, q)$ be the set of unipotent elements in $GL(\alpha, q)$. We define

$$r_\alpha(q) = \frac{1}{|GL(\alpha, q)|} \left|\{(u, x) \in U(\alpha, q) \times R(Q, \alpha, q) \mid ux = x\}\right|.$$

By partitioning $U(q, \alpha)$ into conjugacy classes one easily shows that r_α is a rational function of q (but in general not a polynomial). In this section we prove the following result.

PROPOSITION 2.1. *$r_\alpha(q)$ has no pole in* 0. *Furthermore one has*

$$\sum_{\substack{\beta, \gamma \in \mathbb{N}^{Q_0} \\ \beta + \gamma = \alpha}} r_\beta(0) o_\gamma(0) = \delta_{0\alpha}. \tag{2.1}$$

Proof. The case $\alpha = 0$ is trivial, so we consider the case $\alpha \neq 0$.

By the Burnside formula we have the following expression for $o_\alpha(q)$:

$$o_\alpha(q) = \frac{1}{|GL(\alpha, q)|} \sum_{x \in R(Q, \alpha, q)} |\mathrm{Aut}(V_x)|. \tag{2.2}$$

We will study a slightly modified version of (2.2),

$$t_\alpha(q) = \frac{1}{|GL(\alpha, q)|} \sum_{x \in R(Q, \alpha, q)} |\mathrm{End}(V_x)|. \tag{2.3}$$

Giving an element g of $\mathrm{End}(V_x)$ amounts to giving a decomposition of $V_x = V_{1,x} \oplus V_{2,x}$, $g = (g_1, g_2)$ such that g_1 acts nilpotently on $V_{1,x}$ and $g_2 \in \mathrm{Aut}(V_{2,x})$. Denote by $\mathrm{Nil}(V)$ the nilpotent endomorphisms of a representation V. We find

$$t_\alpha(q) = \frac{1}{|GL(\alpha, q)|} \sum_{\beta + \gamma = \alpha} \psi_{\beta, \gamma} \sum_{\substack{y \in R(Q, \beta, q) \\ z \in R(Q, \gamma, q)}} \left|\mathrm{Nil}(V_y)\right| \cdot \left|\mathrm{Aut}(V_z)\right|.$$

Here $\psi_{\beta, \gamma} = \prod_{i \in Q_0} \psi_{\beta_i, \gamma_i}$ and ψ_{β_i, γ_i} is equal to the number of decompositions $k^{\alpha_i} = B \oplus C$ where B and C are, respectively, vector spaces of dimension β_i and γ_i. It is elementary to see that

$$\psi_{\beta_i, \gamma_i} = \frac{|GL(\alpha_i, q)|}{|GL(\beta_i, q)| \cdot |GL(\gamma_i, q)|}.$$

Substituting yields

$$t_\alpha(q) = \sum_{\beta+\gamma=\alpha} \left(\frac{1}{|GL(\beta, q)|} \sum_{y \in R(Q, \beta, q)} \left|\mathrm{Nil}(V_y)\right| \right) \cdot \left(\frac{1}{|GL(\gamma, q)|} \sum_{z \in R(Q, \gamma, q)} |\mathrm{Aut}(V_z)| \right).$$

Thus we find

$$t_\alpha(q) = \sum_{\beta+\gamma=\alpha} r_\beta(q) o_\gamma(q) \tag{2.4}$$

(where we have used that one can go from a nilpotent endomorphism to a unipotent endomorphism and back by adding and substracting the identity endomorphism). So in particular t_α is a rational function (which could have been shown directly).

Now we consider a different way of evaluating t_α. Let $V_1, \ldots, V_n$ represent the isomorphism classes of representations of Q over $\mathbb{F}_q$, with dimension vector α. Grouping $R(Q, \alpha, q)$ into $GL(\alpha, q)$ orbits yields

$$t_\alpha(q) = \sum_{i=1}^{n} \frac{|\mathrm{End}(V_i)|}{|\mathrm{Aut}(V_i)|}. \tag{2.5}$$

Now let W be an arbitrary representation of Q over $\mathbb{F}_q$ and let $W = W_1^{\oplus a_1} \oplus \cdots \oplus W_p^{\oplus a_p}$ be its decomposition into indecomposables. Then

$$|\mathrm{End}(W)| = \prod_{i,j} \left|\mathrm{Hom}(W_i, W_j)\right|^{a_i a_j}$$

$$|\mathrm{Aut}(W)| = \prod_l \left|GL_{a_l}(\mathrm{End}(W_l))\right| \cdot \prod_{i \neq j} \left|\mathrm{Hom}(W_i, W_j)\right|^{a_i a_j},$$

whence

$$\frac{|\mathrm{End}(W)|}{|\mathrm{Aut}(W)|} = \prod_l \frac{\left|M_{a_l}(\mathrm{End}(W_l))\right|}{\left|GL_{a_l}(\mathrm{End}(W_l))\right|}.$$

$\operatorname{End}(W_l)$ is a local and its residue field $\mathbb{F}_l$ is a finite extension of $\mathbb{F}_q$. Since $GL_{a_l}(\operatorname{End}(W_l))$ is the unit group of $M_{a_l}(\operatorname{End}(W_l))$, and since this unit group is the inverse image of $GL_{a_l}(\mathbb{F}_l)$ we find that

$$\frac{\left|M_{a_l}(\operatorname{End}(W_l))\right|}{\left|GL_{a_l}(\operatorname{End}(W_l))\right|} = \frac{\left|M_{a_l}(\mathbb{F}_l)\right|}{\left|GL_{a_l}(\mathbb{F}_l)\right|}.$$

It is an easy exercise to show that the right-hand side of the above equation is always divisible by q (as a rational number). We conclude that $|\operatorname{End}(W)|/|\operatorname{Aut}(W)|$ is *always* divisible by q. Hence the same holds for $t_\alpha(q)$. Since this holds for all powers of p it follows that $t_\alpha(q)$ must have a zero in 0 (as a rational function).

Now using (2.4), together with the fact that $o_\alpha(q)$ is a polynomial, we obtain by induction on α that $r_\alpha(q)$ does not have a pole in zero. Now substituting $q = 0$ finishes the proof of the theorem. ∎

We will reformulate the above in terms of generating functions. For $\alpha \in \mathbb{C}^{Q_0}$ let $e(\alpha)$ be a formal exponential. Thus $e(\alpha+\beta) = e(\alpha)e(\beta)$. Then the previous proposition may be written as

$$\left(\sum_\beta r_\beta(0)e(\beta)\right)\cdot\left(\sum_\gamma o_\gamma(0)e(\gamma)\right) = 1. \tag{2.6}$$

Let us now consider the relation between $o_\alpha(q)$ and $i_\alpha(q)$. Since every representation has a unique decomposition into idecomposables, we find

$$o_\alpha(q) = \sum_{\substack{\alpha=\sum r_i\beta_i\\ \beta_i\neq\beta_j}} \prod_i \binom{i_\beta(q)+r_i-1}{r_i}$$

This formula becomes more elegant in terms of generating functions

$$\sum_\alpha o_\alpha(q)e(\alpha) = \frac{1}{\prod_\beta(1-e(\beta))^{i_\beta(q)}}. \tag{2.7}$$

Now combining (2.6) and (2.7), together with the fact that $i_\alpha(0) = a_\alpha(0)$ [12] we find

$$\prod_\alpha (1-e(\alpha))^{a_\alpha(0)} = \sum_\alpha r_\alpha(0)e(\beta). \tag{2.8}$$

Let $\Delta_+ \subset \mathbb{C}^{Q_0}$ be the positive roots associated to the bilinear form $(-,-)$ [13].

For $\alpha \in \Delta_+$ let m_α be the corresponding multiplicity, m_α is determined by the formal identity

$$\prod_{\alpha \in \Delta_+} (1 - e(\alpha))^{m_\alpha} = \sum_{w \in W_Q} \epsilon(w) e(\rho - w(\rho)). \tag{2.9}$$

Comparing (2.8) and (2.9) yields a first reformulation of Conjecture 1.4.

PROPOSITION 2.2. *Conjecture* 1.4 *is equivalent to*

$$r_\alpha(0) = \begin{cases} \epsilon(w) & \textit{if } \alpha = \rho - w\rho \\ 0 & \textit{otherwise}. \end{cases} \tag{2.10}$$

The above proposition can also be obtained from the results in Hua's thesis [10].

Below we fill further simplify the computation of $r_\alpha(0)$. This will yield a proof of Theorem A.

3. SOME REMARKS ON SYMMETRIC FUNCTIONS

Let $\mathscr{P}_n$ be the set of partitions of a natural number n. If $\lambda \in \mathscr{P}_n$ then $s_\lambda(x)$ and $P_\lambda(x;t)$ denote, respectively, the Schur function and the Hall–Littlewood function associated to λ (in an infinite number of variables) [17]. As usual we denote by $l(\lambda)$ the largest i such that $\lambda_i \neq 0$, $|\lambda| = \sum \lambda_i$, and $n(\lambda) = \sum (i-1)\lambda_i$. λ' is the conjugate partition to λ. We also associate with λ its diagram in $\mathbb{N}^2$. The notation $y \in \lambda$ means that y is one of the boxes in this diagram. In that case $h(y)$ is the hook length of y.

The relation between s_λ and P_λ is given by

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(t) P_\mu(x;t), \tag{3.1}$$

where $K_{\lambda\mu}(t)$ are the Kostka–Foulkes polynomials in one variable. For use below it is convenient to introduce the modified version

$$\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(q^{-1}).$$

If μ is a partition of n then below u_μ will denote an arbitrary unipotent element of $GL(n,q)$ corresponding to μ. We let $C(u_\mu)$ be its centralizer.

According to [17, Example III.3.2],

$$P_\mu(q^{-1}, q^{-2}, \dots; q^{-1}) = \frac{q^{n(\mu)}}{|C(u_\mu)|}$$

as a formal power series in q^{-1}.

On the other hand by [17, Example I.3.2] (replacing q by q^{-1} and multiplying by $q^{-|\lambda|}$) we find

$$s_\lambda(q^{-1}, q^{-2}, \dots) = \frac{q^{-|\lambda|-n(\lambda)}}{\prod_{y\in\lambda}(1-q^{-h(y)})} = \frac{q^{n(\lambda')}}{\prod_{y\in\lambda}(q^{h(y)}-1)}.$$

Substituting this in (3.1) yields

$$\sum_\mu \frac{\tilde{K}_{\lambda\mu}(q)}{|C(u_\mu)|} = \frac{q^{n(\lambda')}}{\prod_{y\in\lambda}(q^{h(y)}-1)}.$$

Thus we find that the left-hand side of the above equation does not have a pole in $q = 0$. Furthermore

$$\sum_\mu \frac{\tilde{K}_{\lambda\mu}(q)}{|C(u_\mu)|}\bigg|_{q=0} = \begin{cases} (-1)^n & \text{if } \lambda = (1^n) \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

4. A LEMMA FROM THE REPRESENTATION THEORY OF $GL(n, q)$

By definition a *multipartition* of $\alpha \in \mathbb{N}^{Q_0}$ is a list of partitions $\lambda = (\lambda_i)_{i\in Q_0}$ such that $|\lambda_i| = \alpha_i$. We view λ as a two dimensional list of non-negative integers $(\lambda_{ij})_{ij}$ such that $\sum_j \lambda_{ij} = \alpha_i$. We write $\lambda(j)$ for the element of $\mathbb{N}^{Q_0}$ given by $\lambda(j)_i = \lambda_{ij}$. Denote the set of multipartitions of α by $\mathscr{P}_\alpha$. For $\lambda \in \mathscr{P}_\alpha$, u_λ will be an element of $GL(\alpha, q) = \prod_i GL(\alpha_i, q)$ of the form (u_{λ_i}). We also put

$$f_\lambda(\mu) = \prod_i \tilde{K}_{\lambda_i\mu_i}(q).$$

We view the f_λ as functions from $\mathscr{P}_\alpha$ to $\mathbb{Z}[q]$. The following result is easy to see.

LEMMA 4.1. *Let W be a representation of $GL(\alpha, \mathbb{F}_p)$ over $\mathbb{F}_p$. Then the function $\mu \mapsto q^{\dim_{\mathbb{F}_p} W^{u_\mu}}$ is a linear combination of the functions f_λ with coefficients in $\mathbb{Z}[q]$.*

Proof. Since the $f_\lambda(\mu)$ form an upper triangular matrix with q-powers on the diagonal it is clear that we can express $q^{\dim_{\mathbb{F}_p} W^{u_\mu}}$ as a linear combination of the f_λ with coefficients in $\mathbb{Z}[q^{-1}, q]$.

Now let q be a fixed power of p and let V be the permutation representation of $W \otimes_{\mathbb{F}_p} \mathbb{F}_q$ (over $\mathbb{C}$). Then $q^{\dim W^{u_\mu}} = \mathrm{Tr}(u_\mu, V)$. Now by Green's formula for the irreducible characters of $GL(\alpha, q)$ we can express the values of the character of V at unipotent elements as a $\mathbb{Q}$-linear combination of (products of) Green polynomials [22, p. 135]. Furthermore the denominators of the coefficients are bounded in terms of α. Expressing these Green polynomials further in terms of the f_λ [17, III.(7.11)] we find that for a fixed $q, q^{\dim W^{u_\mu}}$ is a linear combination of the $f_\lambda(q)$ with coefficients in $\mathbb{Q}$, whose denominator is still bounded in terms of α.

If we now let q go to ∞ we see that if we consider q as a variable again, the remark of the first paragraph yields that the coefficients must be in $\mathbb{Z}[q]$. ∎

5. PROOF OF THEOREM A

In this section we use all the notations from the previous sections. In particular Q is a fixed quiver without oriented cycles. It is easy to see that we have

$$r_\alpha(q) = \sum_{\lambda \in \mathscr{P}_\alpha} \frac{\left|R(Q, \alpha, q)^{u_\lambda}\right|}{|C(u_\lambda)|}.$$

Now by Lemma 4.1 we have

$$\left|R(Q, \alpha, q)^{u_\lambda}\right| = \sum_\mu c_\mu f_\mu(\lambda),$$

where $c_\mu \in \mathbb{Z}[q]$. This yields

$$r_\alpha(q) = \sum_{\lambda, \mu \in \mathscr{P}_\alpha} c_\mu \frac{f_\mu(\lambda)}{|C(u_\lambda)|}.$$

Let us rewrite this using the definition of f_μ,

$$r_\alpha(q) = \sum_{\mu \in \mathscr{P}_\alpha} c_\mu \prod_{i \in Q_0} \sum_{\lambda_i \in \mathscr{P}_{\alpha_i}} \frac{\tilde{K}_{\mu_i, \lambda_i}(q)}{|C(u_{\lambda_i})|}.$$

Using (3.2) we now find

$$r_\alpha(0) = (-1)^{\sum_i \alpha_i} c_{1^\alpha}(0), \tag{5.1}$$

where $(1^\alpha)_i = 1^{\alpha_i}$. Hence the key point is to find c_{1^α}. Going back to the definition of f_μ we have that

$$\left|R(Q,\alpha,q)^{u_\lambda}\right| = \sum_\mu c_\mu \prod_{i\in Q_0} q^{n(\lambda_i)} K_{\mu_i\lambda_i}(q^{-1}).$$

Multiplying with $\prod_{i\in Q_0} q^{-n(\lambda_i)} K^{-1}_{\lambda_i 1^{\alpha_i}}(q^{-1})$ and summing over λ yields

$$c_{1^\alpha} = \sum_\lambda \left|R(Q,\alpha,q)^{u_\lambda}\right| \prod_{i\in Q_0} q^{-n(\lambda_i)} K^{-1}_{\lambda_i 1^{\alpha_i}}(q^{-1}).$$

Substituting into (5.1) and combining with (1.2) (which is proved in Appendix A) we find

$$(-1)^{\sum_i \alpha_i} c_{1^\alpha} = \sum_{\lambda\in\mathscr{P}_\alpha} (-1)^{\sum_i \lambda'_{i1}} q^{-e_\lambda} \prod_i [\lambda'_i](q^{-1}),$$

where

$$\begin{aligned} e_\lambda &= -\dim R(Q,\alpha,q)^{u_\lambda} + \sum_{i\in Q_0} n(\lambda_i) + \sum_{\substack{i\in Q_0\\ j\geq 2}} \frac{\lambda'_{ij}(\lambda'_{ij}+1)}{2} \\ &= -\dim R(Q,\alpha,q)^{u_\lambda} + \sum_{\substack{i\in Q_0\\ j\geq 0}} \lambda'^2_{ij} - \sum_{i\in Q_0} \frac{\lambda'_{i1}(\lambda'_{i1}+1)}{2}. \end{aligned}$$

On the other hand it is easy to see that

$$\dim R(Q,\alpha,q)^{u_\lambda} = \sum_{\substack{i,k\in Q_0\\ j\geq 0}} a_{ik} \lambda'_{ij} \lambda'_{kj},$$

where a_{ik} is the number of arrows going from i to k. Define $\lambda(j)$ as the element of $\mathbb{N}^{Q_0}$ given by $\lambda(j)_i = \lambda_{ij}$. Then the expression for e_λ can be further rewritten as

$$e_\lambda = \frac{1}{2}\sum_{j\geq 0} (\lambda'(j), \lambda'(j)) - \sum_{i\in Q_0} \frac{\lambda'_{i1}(\lambda'_{i1}+1)}{2}.$$

So now our final expression for $(-1)^{\sum_i \alpha_i} c_{1^\alpha}$ becomes

$$\sum_{\lambda \in \mathscr{P}_\alpha} q^{-\sum_j (\lambda'(j), \lambda'(j))/2} \prod_{i \in Q_0} (-1)^{\lambda'_{i1}} q^{(\lambda'_{i1}(\lambda'_{i1}+1))/2} [\lambda'_i](q^{-1}). \tag{5.2}$$

We now put $p_\alpha(t) = (-)^{\sum_i \alpha_i} c_{1^\alpha}(t^{-1})$. Invoking (5.1) together with Proposition 2.2 and furthermore replacing λ' by λ and q^{-1} by t finishes the proof of Theorem A.

6. SOME COMBINATORIAL CONSIDERATIONS

As a warm-up we will consider Theorem A in the case that Q is a one-vertex quiver with no loops. Then Theorem A asserts that

$$p_a(t) = \sum_{|\lambda| = a} (-1)^{\lambda_1} t^{\sum_i \lambda_i^2 - \lambda_1(\lambda_1+1)/2} [\lambda] \tag{6.1}$$

is a polynomial in t^{-1}. On the other hand inspection reveals that if $a > 1$ then $p_a(t) \in t\mathbb{Z}[t]$. Combining this we obtain:

PROPOSITION 6.1. *The Laurent polynomial $p_a(t)$ given by* (6.1) *is identically zero*.

This result is probably known in some form. In any case the authors found it a pleasant exercise to prove it directly.

Now let us consider the m-arrow quiver. In that case we obtain using the same method as above:

PROPOSITION 6.2. *The Laurent polynomial $p_{a,b}(t)$ defined by* (1.1) *is identically zero if $a \gg b$.*

Computer computations show that probably the following conjecture is true.

Conjecture 6.3. The Laurent polynomial $p_{a,b}(t)$ is identically zero if and only if $a \geq mb + 2$ or $b \geq ma + 2$.

It is clear that a similar reasoning can be applied to more complicated quivers.

APPENDIX A: ON THE INVERSE OF THE KOSTKA–FOULKES MATRIX

In this Appendix we prove formula (1.2). As was already pointed out in the Introduction, this result was also proved by Macdonald [3] in the special case that $\mu = 1^n$.

Denote by $K^{-1}_{\lambda\mu}(t)$ the inverse of the Kostka–Foulkes matrix $K_{\lambda\mu}(t)$. Thus by definition

$$P_\lambda(x;t) = \sum_\mu K^{-1}_{\lambda\mu}(t) s_\mu(x).$$

Our basic tool will be the fact that both P_μ and s_μ satisfy a version of Pieri's formula for multiplication by $s_{1^m} = P_{1^m} = e_m$ (the mth elementary symmetric function [17]). This will eventually lead to a recursion formula for the entries of $K^{-1}(t)$.

Let us use the notation $\mu <_m \nu$ to signify that $\nu - \mu$ is vertical strip of length m (a vertical strip is a skew diagram such that every horizontal line cuts the diagram at most once).

Pieri's formula for s_μ is classical [17, I.(5.17)]:

$$s_\mu e_m = \sum_{\mu <_m \nu} s_\nu. \tag{A.1}$$

Pieri's formula for P_μ is similar [17, III.3],

$$P_\mu e_m = \sum_{\mu <_m \nu} f^\nu_{\mu,1^m}(t) P_\nu \tag{A.2}$$

where

$$f^\nu_{\mu,1^m}(t) = \prod_i \begin{bmatrix} \nu'_i - \nu'_{i+1} \\ \nu'_i - \mu'_i \end{bmatrix}.$$

Here and below $\left[{a \atop b}\right]$ is an abbreviation $\left[{\ \ a \atop b \ \ a-b}\right]$. For simplicity we will routinely use the convention that a Gaussian binomial coefficient $\left[{a \atop b}\right]$ is zero if $b < 0$. This allows us to be somewhat informal with regard to summation bounds.

Substituting $P_\nu = \sum_{\alpha \le \nu} K^{-1}_{\nu\alpha}(t) s_\alpha$, $P_\mu = \sum_{\beta \le \mu} K^{-1}_{\mu\beta}(t) s_\beta$ in (A.2) yields

$$\sum_{\beta \le \mu} K^{-1}_{\mu\beta}(t) s_\beta e_m = \sum_{\mu <_m \nu} \sum_{\alpha \le \nu} f^\nu_{\mu,1^m}(t) K^{-1}_{\nu\alpha}(t) s_\alpha.$$

Using Pieri's formula for s_β now yields

$$\sum_{\beta \le \mu} \sum_{\beta <_m \delta} K^{-1}_{\mu\beta}(t) s_\delta = \sum_{\mu <_m \nu} \sum_{\alpha \le \nu} f^\nu_{\mu,1^m}(t) K^{-1}_{\nu\alpha}(t) s_\alpha.$$

Equating the coefficients of s_δ in this equation we obtain for all pairs of partitions μ, δ such that $|\delta| = m + |\mu|$ a relation

$$\sum_{\beta \le \mu} \sum_{\beta <_m \delta} K^{-1}_{\mu\beta}(t) = \sum_{\mu <_m \nu} \sum_{\delta \le \nu} f^\nu_{\mu,1^m}(t) K^{-1}_{\nu\delta}(t). \tag{A.3}$$

It is easy to see that this relation determines $K_{\psi\epsilon}^{-1}(t)$ uniquely. To see this apply (A.3) with $\delta = \epsilon$ and μ equal to ψ minus the last column. Then (A.3) expresses $K_{\psi\epsilon}^{-1}(t)$ in terms of $K_{\mu\beta}^{-1}(t)$ with $|\mu| < |\psi|$ and $K_{\nu\epsilon}^{-1}$ with $\nu < \psi$. All of these may be assumed to be known by induction.

Equation (A.3) simplifies considerably if we apply it in the case $\delta = (c, 1^d)$. We find

$$K^{-1}_{\mu,(c-1,1^{d-m+1})}(t) + K^{-1}_{\mu,(c,1^{d-m})}(t) = \sum_{\mu <_m \nu} f^{\nu}_{\mu,1^m}(t) K^{-1}_{\nu,(c,1^d)}(t). \quad \text{(A.4)}$$

It follows that in order to prove our theorem, it is sufficient to show that (1.2) gives the correct value if $\lambda = 1^n$ and satisfies the recursion relation (A.4). The first statement is obvious, so we will prove the second one. We will do this in the case $c > 1$. The case $c = 1$ is similar but requires fewer steps.

So assume $c > 1$. Making the appropriate substitutions and replacing the partitions μ, ν by their conjugate ones we find that we have to prove the identity

$$\begin{aligned}
&\left((-1)^{\mu_1+d-m+2} t^{\sum_{i\ge 2}(\mu_i(\mu_i+1))/2-\sum_{j=2}^{c-1}\mu_j} \frac{1-t^{\mu_{c-1}}}{1-t^{\mu_1}} + (-1)^{\mu_1+d-m+1}\right.\\
&\qquad\left.\times t^{\sum_{i\ge 2}(\mu_i(\mu_i+1))/2-\sum_{j=2}^{c}\mu_j} \frac{1-t^{\mu_c}}{1-t^{\mu_1}}\right) \times \begin{bmatrix} & \mu_1 & \\ \mu_1-\mu_2 & \mu_2-\mu_3 & \cdots \end{bmatrix}\\
&= \sum_{\mu <_m \nu} (-1)^{\nu_1+d+1} t^{\sum_{i\ge 2}(\nu_i(\nu_i+1))/2-\sum_{j=2}^{c}\nu_j} \begin{bmatrix} & \nu_1 & \\ \nu_1-\nu_2 & \nu_2-\nu_3 & \cdots \end{bmatrix}\\
&\qquad \times \frac{1-t^{\nu_c}}{1-t^{\nu_1}} \begin{bmatrix} \nu_1-\nu_2 \\ \nu_1-\mu_1 \end{bmatrix} \begin{bmatrix} \nu_2-\nu_3 \\ \nu_2-\mu_2 \end{bmatrix} \cdots .
\end{aligned}$$

We now put $b = \mu_1$, $a_i = \mu_{i-1} - \mu_i$, $r_i = \nu_i - \mu_i$. After some algebraic manipulation the previous equation becomes

$$\begin{aligned}
&(-1)^{-m+1} t^{\sum_{i\ge 2}(\mu_i(\mu_i+1))/2-\sum_{j=2}^{c-1}\mu_j}(2 - t^{\mu_{c-1}} - t^{\mu_c})\\
&\quad = (1-t^{\mu_c}) \sum_{\sum_i r_i = m} (-1)^{r_1} t^{\sum_{i\ge 2}((\mu_i+r_i)(\mu_i+r_i+1))/2-\sum_{j=2}^{c}(\mu_i+r_i)}\\
&\qquad\quad \times \begin{bmatrix} b+r_1-1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots\\
&\qquad\quad + t^{\mu_c}(1-t^{a_c}) \sum_{\sum r_i = m} (-)^{r_1} t^{\sum_{i\ge 2}((\mu_i+r_i)(\mu_i+r_i+1))/2-\sum_{j=2}^{c}(\mu_i+r_i)}\\
&\qquad\qquad \times \begin{bmatrix} b+r_1-1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots \begin{bmatrix} a_c-1 \\ r_c-1 \end{bmatrix} \cdots . \qquad \text{(A.5)}
\end{aligned}$$

Before we continue we derive an identity between Gaussian binomial coefficients using non-commutative generating functions.

If x, y are variables satisfying $yx = txy$ then it is well known that

$$(x+y)^a = \sum_{u \geq 0} \begin{bmatrix} a \\ u \end{bmatrix} x^u y^{a-u}.$$

We will also apply this with negative exponents:

$$\begin{aligned}(x+y)^{-a} &= \sum_{u \geq 0} \begin{bmatrix} -a \\ u \end{bmatrix} x^u y^{-a-u} \\ &= \sum_{u \geq 0} (-1)^u t^{-au+(u(u-1))/2} \begin{bmatrix} a+u-1 \\ u \end{bmatrix} x^u y^{-a-u}. \qquad \text{(A.6)}\end{aligned}$$

Now we introduce variables $(x_i)_i, (y_i)_i$ satisfying

$$\begin{aligned} y_i x_j &= t x_j y_i \\ y_i y_j &= y_j y_i \\ x_i x_j &= x_j x_i \end{aligned}$$

and for $b \in \mathbb{N}$, $(a_i)_i \in \mathbb{Z}$, $a_i = 0$ for $i \gg 0$, we consider

$$\cdots (x_n + y_n)^{a_n} \cdots (x_2 + y_2)^{a_2} (x_1 + y_1)^{-b-1}. \qquad \text{(A.7)}$$

This yields

$$\begin{aligned}\text{(A.7)} &= \sum_{(r_i)_i} (-1)^{r_1} t^{-(b+1)r_1 - (r_1(r_1-1))/2} \begin{bmatrix} b+r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots \\ &\quad \times \cdots x_3^{r_3} y_3^{a_3 - r_3} x_2^{r_2} y_2^{a_2 - r_2} x_1^{r_1} y_1^{-b-1-r_1} \\ &= \sum_{(r_i)_i} (-1)^{r_1} t^{S_r} \begin{bmatrix} b+r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots \times x_1^{r_1} x_2^{r_2} x_3^{r_3} \cdots \\ &\quad \times y_1^{-b-1-r_1} y_2^{a_2 - r_2} y_3^{a_3 - r_3} \cdots,\end{aligned}$$

where

$$S_r = -br_1 - \frac{r_1(r_1+1)}{2} + \sum_{1 \leq j < i} (a_i - r_i) r_j.$$

Substituting $x_i = x$, $y_i = y$ yields

$$\begin{aligned}&(x+y)^{a_2 + a_3 + \cdots - b - 1} \\ &\quad = \sum_{(r_i)_i} (-1)^{r_1} t^{S_r} \begin{bmatrix} b+r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots \times x^{\Sigma r_i} y^{-b-1-\Sigma r_i + \Sigma_{i \geq 2} a_i}.\end{aligned}$$

We consider the case $b = \sum_{i \geq 2} a_i$ and we look at the coefficient of $x^m y^{-m-1}$. Using

$$(x+y)^{-1} = \sum_m (-1)^m t^{-(m(m+1))/2} x^m y^{-m-1}$$

we find

$$\sum_{\sum_i r_i = m} (-1)^{r_1} t^{T_r} \begin{bmatrix} b + r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots = (-1)^m, \tag{A.8}$$

where

$$T_r = S_r + \frac{m(m+1)}{2} = \sum_{j > i \geq 2} r_i a_j + \sum_{i \geq 2} \frac{r_i(r_i+1)}{2}.$$

Let $\mu, \nu, b, a_2, a_3, \ldots$ be related as above. Then a straightforward computation yields that

$$T_r = \sum_{i \geq 2} \frac{\nu_i(\nu_i + 1)}{2} - \sum_{i \geq 2} \frac{\mu_i(\mu_i + 1)}{2}$$

so that we obtain the identity

$$\sum_{\sum_i r_i = m} (-1)^{r_1} t^{\sum_{i \geq 2}((\mu_i + r_i)(\mu_i + r_i + 1))/2} \begin{bmatrix} b + r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots$$
$$= (-1)^m t^{\sum_{i \geq 2}(\mu_i(\mu_i+1))/2}. \tag{A.9}$$

Note that in the proof of this identity we have not used that μ is a partition. In fact μ can be an arbitrary sequence of integers, zero in high degree and positive in degree 1 (the restriction $\mu_1 \geq 0$ comes from the fact that (A.6) is not valid for negative a). Below we will use (A.9) for μ which are not necessarily partitions.

We will use (A.9) to simplify the right-hand side of (A.5). Let's first work on the second term. Put

$$\bar{\mu}_i = \begin{cases} \mu_i - 1 & i < c \\ \mu_i & i \geq c, \end{cases}$$

$$\bar{r}_i = \begin{cases} r_i - 1 & i = c \\ r_i & i \neq c. \end{cases}$$

We make the companion definitions

$$\bar{a}_i = \overline{\mu}_{i-1} - \overline{\mu}_i = \begin{cases} a_i & i \neq c \\ a_i - 1 & i = c, \end{cases}$$

$$\bar{b} = \overline{\mu}_1 = b - 1.$$

Making these substitutions, the second term of the right-hand side of (A.5) becomes

$$t^{\mu_c}(1 - t^{a_c}) \sum_{\Sigma \bar{r}_i = m-1} (-1)^{\bar{r}_1} t^{\Sigma_{i \geq 2}((\overline{\mu}_i + \bar{r}_i)(\overline{\mu}_i + \bar{r}_i + 1))/2} \begin{bmatrix} \bar{b} + \bar{r}_1 \\ \bar{r}_1 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{r}_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ \bar{r}_2 \end{bmatrix} \cdots . \tag{A.10}$$

Using the identity (A.9), (A.10) may be rewritten as

$$(-1)^{m-1} t^{\mu_c}(1 - t^{a_c}) t^{\Sigma_{i \geq 2}(\overline{\mu}_i(\overline{\mu}_i + 1))/2},$$

which is equal to

$$(-1)^{m-1} t^{\mu_c}(1 - t^{a_c}) t^{\Sigma_{i \geq 2}(\mu_i(\mu_i + 1))/2 - \Sigma_{j=2}^{c-1} \mu_j}.$$

Substracting this from the left-hand side of (A.5), and dividing by $1 - t^{\mu_c}$, we are left with proving

$$\begin{aligned} &(-1)^m t^{\Sigma_{i \geq 2}(\mu_i(\mu_i + 1))/2 - \Sigma_{j=2}^{c} \mu_j}(1 - t^{\mu_c}) \\ &\quad = \sum_{\Sigma r_i = m} (-1)^{r_1} t^{\Sigma_{i \geq 2}((\mu_i + r_i)(\mu_i + r_i + 1))/2 - \Sigma_{j=2}^{c}(\mu_j + r_j)} \\ &\quad\quad \times \begin{bmatrix} b + r_1 - 1 \\ r_1 \end{bmatrix} \begin{bmatrix} a_2 \\ r_2 \end{bmatrix} \begin{bmatrix} a_3 \\ r_3 \end{bmatrix} \cdots . \end{aligned} \tag{A.11}$$

We now work on the right-hand side of (A.11). We first make the change of variables

$$\overline{\mu}_i = \begin{cases} \mu_i - 1 & i < c \\ \mu_i & i \geq c, \end{cases}$$

and companion definitions

$$\bar{a}_i = \overline{\mu}_{i-1} - \overline{\mu}_i = \begin{cases} a_i & i \neq c \\ a_i - 1 & i = c, \end{cases}$$

$$\bar{b} = \overline{\mu}_1 = b - 1.$$

So the right-hand side of (A.11) becomes

$$\sum_{\sum r_i = m} (-)^{r_1} t^{\sum_{i\geq 2}((\bar{\mu}_i + r_i)(\bar{\mu}_i + r_i + 1))/2 - (\bar{\mu}_c + r_c)} \begin{bmatrix} b + r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ r_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{a}_c + 1 \\ r_c \end{bmatrix} \cdots .$$

Now we use the identity

$$\begin{bmatrix} \bar{a}_c + 1 \\ r_c \end{bmatrix} = \begin{bmatrix} \bar{a}_c \\ r_c - 1 \end{bmatrix} + t^{r_c} \begin{bmatrix} \bar{a}_c \\ r_c \end{bmatrix}$$

and we obtain that the right-hand side of (A.11) is equal to

$$\sum_{\sum r_i = m} (-)^{r_1} t^{\sum_{i\geq 2}((\bar{\mu}_i + r_i)(\bar{\mu}_i + r_i + 1))/2 - (\bar{\mu}_c + r_c)} \begin{bmatrix} \bar{b} + r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ r_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{a}_c \\ r_c - 1 \end{bmatrix} \cdots$$
$$+ \sum_{\sum r_i = m} (-)^{r_1} t^{\sum_{i\geq 2}((\bar{\mu}_i + r_i)(\bar{\mu}_i + r_i + 1))/2 - \bar{\mu}_c} \begin{bmatrix} \bar{b} + r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ r_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{a}_c \\ r_c \end{bmatrix} \cdots .$$

Using (A.9) the second part of the previous formula is seen to be equal to

$$(-1)^m t^{-\bar{\mu}_c + \sum_{i\geq 2}(\bar{\mu}_i(\bar{\mu}_i + 1))/2} = (-1)^m t^{\sum_{i\geq 2}(\mu_i(\mu_i + 1))/2 - \sum_{j=2}^{c} \mu_j}.$$

Substracting this from the left-hand side of (A.11) we are now left with proving

$$\begin{aligned} &(-)^{m+1} t^{\sum_{i\geq 2}(\mu_i(\mu_i+1))/2 - \sum_{j=2}^{c-1} \mu_j} \\ &\quad = \sum_{\sum r_i = m} (-)^{r_1} t^{\sum_{i\geq 2}((\bar{\mu}_i + r_i)(\bar{\mu}_i + r_i + 1))/2 - (\bar{\mu}_c + r_c)} \\ &\quad\quad \times \begin{bmatrix} \bar{b} + r_1 \\ r_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ r_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{a}_c \\ r_c - 1 \end{bmatrix} \cdots . \end{aligned} \tag{A.12}$$

We now put

$$\bar{r}_i = \begin{cases} r_i & i \neq c \\ r_c - 1 & i = c. \end{cases}$$

Then the right-hand side of (A.12) becomes

$$\sum_{\bar{r}_i = m-1} (-1)^{\bar{r}_1} t^{\sum_{i\geq 2}((\bar{\mu} + \bar{r}_i)(\bar{\mu} + \bar{r}_i + 1))/2} \begin{bmatrix} \bar{b} + \bar{r}_1 \\ \bar{r}_1 \end{bmatrix} \begin{bmatrix} \bar{a}_2 \\ \bar{r}_2 \end{bmatrix} \cdots ,$$

which by (A.9) is equal to

$$(-1)^{m-1} t^{\Sigma_{i \geq 2}(\overline{\mu}_i(\overline{\mu}_i+1))/2} = (-1)^{m-1} t^{\Sigma_{i \geq 2}(\mu_i(\mu_i+1))/2 - \Sigma_{j=2}^{c-1} \mu_j}.$$

This is indeed equal to the left-hand side of (A.12) and hence we are done.

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